AD/A-005 265

MATHEMATICAL MODELLING OF THE ROYAL AIRCRAFT ESTABLISHMENT (RAE) LIGHT-WEIGHT FLEXIBLE SOLAR ARRAY

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20000727320

Prepared for:

Defence Research Information Centre

October 1974

ROYAL AIRCRAFT ESTABLISHMENT

TECHNICAL REPORT 74112

OCTOBER 1974

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PROCUREMENT EXECUTIVE, MINISTRY OF DEFENCE
FARNBOROUGH, HANTS

UDC 629.19.066.5 : 5.001.57

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Received for printing 9 August 1974

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SUMMARY

The paper describes the development of a mathematical model for the attitude dynamics of a spacecraft equipped with a lightweight flexible solar array of RAE design. The theory has been developed using a continuous mechanics approach and a computer programme prepared to generate the lateral bending modes of a spacecraft comprising a rigid central structure carrying a pair of solar arrays symmetrically situated about the central body. Furthermore the programme will generate the effective inertia and mass as a function of the forcing frequency. These are then formulated in the form of transfer functions which are more convenient for control problem analysis.

Two models are considered. The first model assumes that the array cross members are rigid whilst the second takes account of the flexibility of the cross members.

Paper presented at the VI IFAC Symposium "Control in Space", Tsakhkadzor, Armenia, USSR, 26-31 August 1974.

Departmental Reference: Space 465

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Neg.C11608-C11609

1 INTRODUCTION

As the power requirements for future generations of communication and scientific satellites continue to escalate, radically different design departures must be made from the early spinning drum type of satellite with body mounted solar arrays. These designs have already included rigid fold-up arrays (Skylab), semi-rigid fold-up arrays (Boeing Mars mission), flexible roll-up arrays (FRUSA), flexible fold-up arrays using BISTEMS (CTS) and flexible fold-up arrays using a central telescopic mast (X4) and are usually sun orientated from a three axis stabilised spacecraft. Designs embodying flexible panels have been shown [1] to be superior to rigid and semi-rigid types in terms of power-to-weight ratio, stowed volume, adaptability and development potential. The estimated end-of-life power to weight ratios of a two paddle 2kW array of the thinnest available solar cells, for instance, complete with orientation and power transfer mechanisms, range from 20 W kg^{-1} for a rigid fold-up type to 43 W kg^{-1} for a flexible fold-up type deployed by a pneumatic telescopic mast. At the RAE (Space Department) effort has been concentrated into the development of the latter type of lightweight flexible array [2,3].

Fig.1 is an artist's impression of the RAE type array fully deployed, as originally proposed for a spacecraft capable of demonstrating orbital manoeuvres by electric propulsion whilst Fig.2 illustrates the main features of the RAE prototype which has been developed over the past six years and has now satisfactorily completed qualification tests for geostationary orbit.

Due to the lightweight structure of the array, the design of an attitude control system for a spacecraft of the type illustrated in Figs.1 and 2 has to take into account the flexible nature of the array since the natural vibration modes of the spacecraft may be excited by the control torques and forces applied to the spacecraft for the purpose of attitude control, station keeping and orbit manoeuvres. The present paper describes a method which has been employed to assess the various modes of vibration.

2. BRIEF DESCRIPTION OF THE RAE ARRAY

Each of the two arms of the deployed array consist of a number of panels (six pairs in the current design) of thin (100 μm) silicon cells mounted on flexible Kapton sheets supported by aluminium honeycomb cross members attached to a pneumatically deployed telescopic mast. The mast is made up of a number of thin walled aluminium alloy tubes of progressively decreasing diameter and wall thickness, adjacent sections of tube being locked together at the overlaps to give a fairly rigid connection. The cross members are attached to the outboard ends of the tubular sections and the panels are held under tension by springs connected to adjacent cross members. During launch each arm of the array is folded concertina fashion and stowed in an aluminium honeycomb compartment, the cross member on the outermost tube forming the cover of the compartment. Once deployed in orbit, the present design of array cannot be retracted.

3 MODELLING ASSUMPTIONS FOR THE ARRAY

Although such arrays are frequently investigated nowadays by using the method of finite elements in the final phase of analysis, this can be expensive and time consuming in the initial phase of design. The approach employed in this paper has been to use the traditional methods of continuum mechanics rather than a discrete method since the geometry of the system is relatively simple and the method can give a quick insight into the problems involved. This method has also been successfully used by Hughes [4,5,6].

Although torsional and in-plane vibrations will occur, initial estimates showed that these modes of vibration are at much higher frequencies than the dominating out-of-plane modes. Consequently only the lateral flapping modes of vibration are considered in this paper.

In order to model the system, certain idealisations have been made:-

(i) the sections of the mast have been treated as uniform cylindrical beams of specified stiffness and density under compression due to the panels and the overlap portions of the mast neglected apart from their mass contribution. Rotary inertia and shear are neglected.

(ii) the panels have been idealised as uniform membranes of specified density under uniform tension but with negligible flexural rigidity.

(iii) the cross members have been assumed rigid in the first instance so that they can be considered as point masses possessing inertia. Flexible cross members are considered later in the paper.

(iv) the central body is assumed to be rigid with the arrays being deployed symmetrically about the centre of mass.

(v) small linear displacements and zero energy dissipation is assumed, i.e. zero damping.

4 ANALYSIS OF THE LATERAL MOTION

Fig.3 shows the coordinate system used to describe the mode shape of the idealised array at the ith section ($i = 1$ to N). [The index i will be omitted wherever possible in the text to avoid unnecessary symbolic complexities.] The quantity $u(y)$ denotes the mast displacement from a fixed reference plane whilst $v(y)$ represents the corresponding panel displacement.

Since the panels are under tension T and each mast section under the same compression force, the equations governing their motion are respectively

$$Tv'' + \mu\ddot{v} = 0 \qquad (1)$$

and

$$\alpha u'''' + Tu'' + \rho\ddot{u} = 0 \qquad (2)$$

(using the notation prime to signify partial differentiation with respect to y and a dot to denote partial differentiation with respect to time)

where μ = mass per unit length of the panel
ρ = mass per unit length of the mast
E = Young's modulus of the mast
I = area moment of inertia of mast = $\frac{\pi}{4}\left(r_1^4 - r_2^4\right)$
α = EI
r_1 = outer radius of the mast = $r_2 + h$
r_2 = inner radius of the mast
h = wall thickness of the mast
ℓ = length of the mast section.

Thus, when the system is vibrating with sinusoidal frequency p, the solution to the panel equation (1) is

$$v(y) = \frac{u(\ell)\sin ky + u(0)\sin k(\ell - y)}{\sin k\ell} \tag{3}$$

where $u(0)$ and $u(\ell)$ denote the mast displacement at the beginning and end of the section respectively and $k = p\sqrt{\frac{\mu}{T}}$. Similarly, it may be shown [7] that the mast displacement is given by

$$\begin{aligned}(\delta^2 + \epsilon^2)u(y) = {} & (\delta^2 \cos \epsilon y + \epsilon^2 \cosh \delta y)u_0 + \left(\frac{\delta^2 \sin \epsilon y}{\epsilon} + \frac{\epsilon^2 \sinh \delta y}{\delta}\right)u_0' \\ & + \frac{1}{\alpha}(\cosh \delta y - \cos \epsilon y)M_0 + \frac{1}{\alpha}\left(\frac{\sin \epsilon y}{\epsilon} - \frac{\sinh \delta y}{\delta}\right)V_0 \end{aligned} \tag{4}$$

where $M = \alpha u''$ is the mast moment and $V = -M'$. The mast shear S is given by $S = V - Tu'$, i.e. it is dependent upon the compression force and the quantities β, γ, δ and ϵ are defined by

$$\begin{aligned} \gamma &= \sqrt{\frac{T}{\alpha}}, & \beta^2 &= p\sqrt{\frac{\rho}{\alpha}}, \\ \delta^2 &= -\frac{\gamma^2}{2} + \sqrt{\frac{\gamma^4}{4} + \beta^4}, & \epsilon^2 &= \frac{\gamma^2}{2} + \sqrt{\frac{\gamma^4}{4} + \beta^4}. \end{aligned} \tag{5}$$

Consequently, using equation (4), we may derive the transition matrix $U^m = [u_{ij}]$ relating the quantities u, u', M and V at the two ends of the mast section, namely

$$\begin{bmatrix} u \\ u' \\ M \\ V \end{bmatrix}_\ell = U^m \begin{bmatrix} u \\ u' \\ M \\ V \end{bmatrix}_0 \tag{6}$$

where the elements of $(\epsilon^2 + \delta^2)U^m$ are given by

$$\begin{bmatrix} \epsilon^2 ch + \delta^2 c & \dfrac{\delta^2 s}{\epsilon} + \dfrac{\epsilon^2 sh}{\delta} & \dfrac{1}{a}(ch - c) & \dfrac{1}{a}\left(\dfrac{s}{\epsilon} - \dfrac{sh}{\delta}\right) \\ \delta\epsilon(\epsilon sh - \delta s) & \epsilon^2 ch + \delta^2 c & \dfrac{1}{a}(\delta sh + \epsilon s) & \dfrac{1}{a}(c - ch) \\ a\delta^2\epsilon^2(ch - c) & a\delta\epsilon(\epsilon sh - \delta s) & \delta^2 ch + \epsilon^2 c & -\epsilon s - \delta sh \\ -a\delta^2\epsilon^2(\delta sh + \epsilon s) & -a\delta^2\epsilon^2(ch - c) & -\delta^3 sh + \epsilon^3 s & \delta^2 ch + \epsilon^2 c \end{bmatrix}$$

using the notation $s = \sin \epsilon \ell$, $c = \cos \epsilon \ell$, $sh = \sinh \delta \ell$ and $ch = \cosh \delta \ell$.

Consider, now, the forces and moments acting at a cross member. These are shown in Fig.4 at the junction between the ith and (i+1)th sections where the suffix notation + and - has been used to denote 'just before' and 'just after' the junction. The total lateral forces $F_\pm$ are equal to the sum of the corresponding shears in the mast $S_\pm$ and the components $Tv'_\pm$ arising from the tension in the panels. Thus

$$F_\pm = S_\pm + Tv'_\pm = V_\pm - Tu'_\pm + Tv'_\pm \quad . \tag{7}$$

The moments acting on the ith cross member arise purely from the neighbouring moments $M_+ \equiv M_{i+1}(0)$ and $M_- = M_i(\ell)$ existing in the mast sections. Thus if we now introduce the state variable $X = (u, u', M, F)$ we may write

$$\begin{bmatrix} u \\ u' \\ M \\ V \end{bmatrix}_0 = U^+ \begin{bmatrix} u \\ u' \\ M \\ F \end{bmatrix}_0 \quad \text{and} \quad \begin{bmatrix} u \\ u' \\ M \\ F \end{bmatrix}_\ell = U^- \begin{bmatrix} u \\ u' \\ M \\ V \end{bmatrix}_\ell \tag{8,9}$$

where the matrices $U^\pm$ are derived by using equations (3), (6) and (7) to be

$$U^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{T'(\cos k\ell - u_{11})}{\Delta} & \dfrac{T - T'u_{12}}{\Delta} & -\dfrac{T'u_{13}}{\Delta} & \dfrac{1}{\Delta} \end{bmatrix} \tag{10}$$

and

$$U^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T'(\cos k\ell - u_{11}) & T'u_{12} - T & -T'u_{13} & \Delta \end{bmatrix} \tag{11}$$

where $T' = Tk/\sin k\ell$ and $\Delta = 1 + T'u_{14}$. Furthermore, the equations of motion for the cross member of mass and moment of inertia m^c and I_c are

$$F_+ - F_- = m^c\ddot{u}$$

and

$$M_+ - M_- = I^c\ddot{u}'$$

giving the transition matrix U^c defined by

$$\begin{aligned} X_+ &= U^c X_- \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -I^c p^2 & 1 & 0 \\ -m^c p^2 & 0 & 0 & 1 \end{bmatrix} X_- \ . \end{aligned} \tag{12}$$

Hence, using the transition matrices defined by equations (6), (8), (9) and (12) the overall transition matrix A for a single array relating the state X_{N+} at the end of the mast to that corresponding to the connection of the mast to the satellite, i.e.

$$X_{N+} = AX_{0-} \tag{13}$$

is

$$A = \prod_{i=1}^{N} \left(U_i^c \, U_i^- \, U_i^m \, U_i^+\right) \ . \tag{14}$$

We will denote the elements of A by a_{ij} $(i,j = 1 \text{ to } 4)$.

5 MODES OF VIBRATION

In view of the symmetry assumptions made concerning the deployed state of the two arrays, the lateral deflections can be considered as the superposition of two classes of normal mode responses. The first class consists of those modes in which the two masts bend in opposite directions (i.e. antisymmetrical deflections) so that there is rotation of the main body but no translation. The second class is comprised of the modes in which the two masts bend in the same direction (i.e. symmetrical deflections) and there is then translation of the central body but no rotation. These will be referred to as the rotational and translational modes respectively. For the attitude control system it is only the rotational modes which are of interest since it is these which affect the pointing accuracy of the satellite. Another class of modes will be called the cantilever modes and refers to the modes of vibration occuring when the satellite is rigidly fixed, i.e. of infinite inertia and mass.

In order to determine the various modes of vibration the appropriate boundary conditions must be applied to the transition equations (13). These will now be considered in turn.

6 ROTATIONAL AND CANTILEVER MODES

If the rotation of the satellite main body, of mass M_b and inertia I_b, is denoted by θ, the arm length from the centre of mass to the array attachment point by d and the applied couple about the centre of mass by C then the boundary conditions at O_- are

$$u = d\theta , \qquad u' = \theta , \qquad I_t\ddot{\theta} = 2M + 2Fd + C$$

and at N_+, $M = F = 0$. From these conditions applied to equation (13) it follows, after some straightforward manipulations, that the effective inertia $I_e = C/\ddot{\theta}$ of the satellite is

$$I_e(p) = I_b - \frac{2N_R}{p^2 D_C} \tag{15}$$

where

$$N_R = \begin{vmatrix} da_{31} + a_{32} & a_{34} - da_{33} \\ da_{41} + a_{42} & a_{44} - da_{43} \end{vmatrix} \tag{16}$$

and

$$D_C = \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \tag{17}$$

are independent of the central body inertia.

Now the natural rotational modes of vibration ω_n are those obtained when the applied couple is zero, i.e. $I_e(\omega_n) = 0$. Also, the cantilever frequencies Ω_n occur when the satellite rotation is zero under an applied couple, i.e. $I_e(\Omega_n)$ is infinite or $D_C(\Omega_n) = 0$. Thus by plotting I_e and D_C as a function of p the rotational and cantilever frequencies may easily be determined. Furthermore, equation (15) shows that the dependence of the rotational frequencies on the central body inertia may be deduced simply by a shift in the I_e ordinate axis. It may also be shown that in the limit as $p \to 0$, $I_e(p)$ tends to the overall inertia I_r of the satellite considered as a rigid body.

7 TRANSLATIONAL MODES OF VIBRATION

Denoting the lateral displacement of the satellite centre of mass by Z and the applied force by P then the boundary conditions at O_- are

$$u = Z , \qquad u' = 0 , \qquad M_b\ddot{Z} = 2F + P$$

whilst at N_+, $M = F = 0$. Again, applying these conditions to equation (13) leads to an effective overall mass $M_e = P/\ddot{Z}$ given by

$$M_e(p) = M_b - \frac{2N_T}{p^2 D_C} \tag{18}$$

where
$$N_T = \begin{vmatrix} a_{33} & a_{31} \\ a_{43} & a_{41} \end{vmatrix} \tag{19}$$

which, as to be expected, is independent of the arm distance d . Furthermore, in the limit as the vibrational frequency tends to zero, $M_e(p)$ tends to the overall rigid mass M_r of the satellite. The natural translational frequencies of vibration ω_n' are those existing when the applied force is zero, i.e. $M_e(\omega_n') = 0$ and, of course, $M_e(\Omega_n)$ is infinite at the cantilever frequencies. As with the rotational natural frequencies, the translational natural frequencies may be determined by plotting M_e as a function of p and noting the zeros. Furthermore, as can be seen from equation (18), the effect of the central body mass on the frequencies may be found again by a corresponding shift in the ordinate axis M_e .

8 TRANSFER FUNCTION REPRESENTATION OF THE SATELLITE

Knowing the natural modes of vibration we may thus express the response to the applied forces and couples in the form of transfer functions, namely

$$\frac{Z}{P} = \frac{1}{M_r s^2} \frac{\prod \left[1 + \frac{s^2}{\Omega_n^2(1 + i\Gamma_n)}\right]}{\prod \left[1 + \frac{s^2}{\omega_n'^2(1 + i\delta_n)}\right]} \tag{20}$$

and

$$\frac{\theta}{C} = \frac{1}{I_r s^2} \frac{\prod \left[1 + \frac{s^2}{\Omega_n^2(1 + i\Gamma_n)}\right]}{\prod \left[1 + \frac{s^2}{\omega_n^2(1 + i\gamma_n)}\right]} \tag{21}$$

where $s = ip$ and structural damping has been introduced in the customary manner [8], i.e. the rigid inertia and mass have modification factors due to the flexibility of the array so that the control block diagram is relatively simple for the particular configuration chosen here (although it may be readily generalised) and only requires knowledge of the natural frequencies. In practice only sufficient factors in equations (20) and (21) would be retained to cover the bandwidth of the control system.

This form is in contrast to the form employing modal gains [4,5,6] which require the actual mode shapes to be calculated and use of their orthogonality properties. The orthogonality conditions for the present model can easily be deduced by consideration of the total kinitic energy of the system and using the 'conjugate property' [9]. In essence, the transfer functions depends upon a mixed representation of the constrained and unconstrained modes of [4,6].

9 COMPUTER PROGRAMME

In order to obtain the effective inertia and mass as a function of frequency a computer programme was written to perform the matrix multiplications indicated by equations (14), (15) and (18) as a function of the basic physical parameters $(T, \mu, E, \rho, r_1, h, \ell)_i$, M_b, I_b and N. The programme was also organised to iterate to the zeros of I_e, M_e and D_C in order to obtain the rotational, translational and cantilever frequencies and, if necessary, to print out the corresponding mode shapes. (Normalised to have unity displacement at the tip of the mast.)

As an illustration of some of the results obtained Figs.5 and 6 show the effective inertia and mass obtained for the array having the physical characteristics shown in Table 1. It will be seen that the rigid inertia of the two arrays is 390 kg m^2 whilst the lowest rotational frequency for a central body inertia of, say, 300 kg m^2 is 0.44 Hz and decreasing to the cantilever frequency of 0.3 Hz as the central body inertia is increased. The behaviour at 0.9 Hz and 1.8 Hz is not shown in detail since there are other very close rotational frequencies corresponding to panel excitation. These rotational frequencies, and their variation with mast stiffness, are shown in Fig.7 for $I_b = 300$ kg m^2.

Table I

Physical Characteristics of Single Array (Mass = 9.20 kg)

Section	ℓ (m)	μ (kg/m)	$\mu\ell$ (kg)	r_1 (mm)	h (mm)	ρ (kg/m)	$\rho\ell$ (kg)	m_c (kg)
1	1.295	0.9594	1.243	25.40	0.4572	0.2180	0.2825	0.0820
2	1.295	0.9594	1.243	25.40	0.4572	0.2180	0.2825	0.0288
3	1.295	0.9594	1.243	22.23	0.4572	0.1766	0.2288	0.0288
4	1.295	0.9594	1.243	20.64	0.4572	0.1623	0.2102	0.0288
5	1.295	0.9594	1.243	19.05	0.3810	0.1250	0.1619	0.0288
6	1.295	0.9594	1.243	17.46	0.3810	0.1143	0.1481	0.2268
Total	7.770		7.458				1.314	0.424

Panel tension 5N, Young's modulus of mast 7.03×10^{10} N m^{-2}, width 1.37 m

10 EXTENSION OF THE MODEL FOR FLEXING CROSS MEMBERS

For weight reduction purposes, the possibility of replacing the stiff cross members by relatively flexible steel hyperdermic tubing is being seriously considered. In order to determine the consequent change in flexibility of the array, another model has been developed which will now be very briefly described. The coordinate system is similar to that of Fig.3 except that the panel motion $v(x,y)$ is now a function of x and y whilst $w(x)$ represents the displacement of the cross member from its undeflected position and is represented by a superposition of clamped-free beam modes $\varphi_i(x)$ [10], i.e. $w(x) = \Sigma q_i \varphi_i(x)$ with $\varphi_i(0) = \varphi_i'(0) = 0$. The φ_i satisfy, in particular, the equations [11]

$$\int_c \varphi_i\varphi_j = c\delta_{ij} \ , \quad \int_c \varphi_i = \frac{4\alpha_i}{\beta_i} = c\alpha_i' \ , \quad \int_c \varphi_i''\varphi_j'' = \beta_i^4 c\delta_{ij} \tag{22}$$

where the suffix c on the integral denotes integration over the length of the cross member and the quantities α_i and $\beta_i c/2$ are tabulated in [10].

In order to develop the equations of motion recourse may be made to Hamilton's principle, namely

$$\delta\int (L + W)dt = 0 \tag{23}$$

in which δ denotes a virtual variation, L is the array Lagrangion and W is the work done on the array, e.g. by the forces and moments at the root of the array. Here the Lagrangian may be written as

$$L = \frac{1}{2}\sum\left[\int_m \rho\dot{u}^2 dy + \int_p \mu\dot{v}^2 dxdy + \int_c \frac{m^c}{c}(\dot{u} + \dot{w})^2 dx - \int_m a u''^2 dy + \int_m Tu'^2 dy - \int_p \frac{T}{c}v'^2 dxdy - \int_c a^c\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\right] \tag{24}$$

where $\sum$ indicates summation over all sections of the array and a^c is the bending stiffness of the cross member.

Carrying out the variation in equation (23) and (24) leads again to the partial differential equations (1) and (2) with solutions (4) and

$$v(x,y) = v_\ell(x)\frac{\sin ky}{\sin k\ell} + v_0(x)\frac{\sin k(\ell - y)}{\sin k\ell} \tag{25}$$

where $v_0(x) = u(0) + \Sigma q_i^0\varphi_i(x)$

$v_\ell(x) = u(\ell) + \Sigma q_i^\ell\varphi_i(x)$

together with the boundary conditions at a cross member

$$S_+ - S_- + \int_c \left(\frac{T}{c}v'\right)_+ dx - \int_c \left(\frac{T}{c}v'\right)_- dx + p^2 m^c(u + \Sigma\alpha_i' q_i) = 0 \tag{26}$$

and

$$\int_c \left(\frac{T}{c}v'\right)_+ \varphi_i dx - \int_c \left(\frac{T}{c}v'\right)_- \varphi_i dx - K_i q_i + p^2 m^c(u\alpha_i' + q_i) = 0 \tag{27}$$

$$(i = 1,2 \ldots)$$

where $K_i = a^c\beta_i^4 c$. Using equation (25), we may evaluate the quantities

$$\int_c \left(\frac{T}{c}\, v'\right)_+ dx \;=\; T'\left[\left(u_\ell + \Sigma\alpha_i' q_i^\ell\right) - \left(u_0 + \Sigma\alpha_i' q_i^\ell\right)\cos k\ell\right] \;\equiv\; G_0^0 \tag{28}$$

$$\int_c \left(\frac{T}{c}\, v'\right)_- dx \;=\; T'\left[\left(u_\ell + \Sigma\alpha_i' q_i^\ell\right)\cos k\ell - \left(u_0 + \Sigma\alpha_i' q_i^\ell\right)\right] \;\equiv\; G_0^\ell \tag{29}$$

$$\int_c \left(\frac{T}{c}\, v'\right)_+ \varphi_i\, dx \;=\; T'\left[\left(u_\ell\alpha_i' + q_i^\ell\right) - \left(u_0\alpha_i' + q_i^0\right)\cos k\ell\right] \;\equiv\; G_i^0 \tag{30}$$

$$\int_c \left(\frac{T}{c}\, v'\right)_- \varphi_i\, dx \;=\; T'\left[\left(u_\ell\alpha_i' + q_i^\ell\right)\cos k\ell - \left(u_0\alpha_i' + q_i^0\right)\right] \;\equiv\; G_i^\ell \tag{31}$$

and equations (26) and (27) become respectively

$$F^+ - F^- + p^2 m^c\left(u + \Sigma q_i\alpha_i'\right) \;=\; 0 \tag{32}$$

and

$$G_i^+ - G_i^- - K_i q_i + p^2 m^c\left(u\alpha_i' + q_i\right) \;=\; 0 \tag{33}$$

$$(i = 1, 2, \ldots)$$

where $F = S + G_0$. Furthermore, we have the equations of continuity at a cross member, namely

$$u^+ = u^- \;,\qquad u'^+ = u'^- \;,\qquad M^+ = M^- \;,\qquad q_i^+ = q_i^- \tag{34}$$

where we have assumed that the moments of inertia of the cross members are negligible.

If we terminate the expansion for $w(x)$ by retaining only n terms then equations (32), (33) and (34) enable us to obtain the transition matrix U^c at a cross member relating the state variable $Y = (q_1 \ldots q_n\; u\; u'\; M\; F\; G_1 \ldots G_n)$ of order $2n + 4$ at ''ust before' to 'just after' the junction, i.e.

$$Y_+ \;=\; U^c Y_-$$

where the elements of U^c are given by

$$\left[\begin{array}{ccc:c:c}
\multicolumn{3}{c:}{1_{n+3}} & 0 & 0_n \\ \hdashline
-p^2m^c\alpha_1' & \cdots & -p^2m^c\alpha_n' & -p^2m^c & \\
(K_1 - p^2m^c) & & & -p^2m^c\alpha_1' & \\
& \ddots & & \vdots & 1_n \\
& & (K_n - p^2m^c) & -p^2m^c\alpha_n' &
\end{array}\right]$$

and 1_n, 0_n denote unity and null matrices of order n respectively.

Using the relation $S = V - Tu'$, equation (6) may be modified to be

$$\begin{bmatrix} u \\ u' \\ M \\ S \end{bmatrix}_\ell = U^{*m} \begin{bmatrix} u \\ u' \\ M \\ S \end{bmatrix}_0 \tag{35}$$

where

$$U^{*m} = U^m + \begin{bmatrix} 0 & Tu_{14} & 0 & 0 \\ 0 & Tu_{24} & 0 & 0 \\ 0 & Tu_{34} & 0 & 0 \\ -Tu_{21} & Tu_{44} - T(u_{22} + Tu_{24}) & -Tu_{23} & -Tu_{24} \end{bmatrix}$$

with elements v_{ij}.

Consider now equations (28), (29), (30), (31), (35) and the relation $F_{0,\ell} = (S + G_0)_{0,\ell}$. These give $3n + 8$ equations relating the $2(2n + 6)$ quantities $(q_1 \ldots q_n\ u\ u'\ M\ S\ F\ G_0\ G_1 \ldots G_n)_{0,\ell}$. By eliminating the four quantities $(S, G_0)_{0,\ell}$ we are left with $(2n + 4)$ equations relating the $2(2n + 4)$ elements of $Y_{0,\ell}$, i.e. a transition equation in the form

$$T_1 Y_\ell = T_0 Y_0 \tag{36}$$

for the ith section of the mast. Besides the $2n$ equations (30) and (31), the other four equations are

$$\Sigma v_{14} T' a_i' q_i^\ell + (1 + v_{14} T') u_\ell = \Sigma v_{14} T' \cos k\ell\, a_i' q_i^0 + (v_{11} + v_{14} T' \cos k\ell) u_0 + v_{12} u_0' + v_{13} M_0 + v_{14} F_0$$

$$\Sigma v_{24} T' a_i' q_i^\ell + v_{24} T' u_\ell + u_\ell' = \Sigma v_{24} T' \cos k\ell\, a_i' q_i^0 + (v_{21} + v_{24} T' \cos k\ell) u_0 + v_{22} u_0' + v_{23} M_0 + v_{24} F_0$$

$$\Sigma v_{34} T' a_i' q_i^\ell + v_{34} T' u_\ell + M_\ell = \Sigma v_{34} T' \cos k\ell\, a_i' q_i^0 + (v_{31} + v_{34} T' \cos k\ell) u_0 + v_{32} u_0' + v_{33} M_0 + v_{34} F_0$$

$$\Sigma (v_{44} - \cos k\ell) T' a_i' q_i^\ell + (v_{44} - \cos k\ell) T' u_\ell + F_1$$
$$= \Sigma (v_{44} \cos k\ell - 1) T' a_i' q_i^0 + (v_{41} + v_{44} T' \cos k\ell - T') u_0 + v_{42} u_0' + v_{43} M_0 + v_{44} F_0$$

so that the elements of T_0 and T_1 are determined explicitly. Consequently, the overall transition matrix B for a single array relating the state at the end of the mast to that at the root, i.e.

$$Y_{N+} = BY_{0-}$$

is

$$B = \prod_{i=1}^{N}\left(U_i^c\, T_{1i}^{-1}\, T_{0i}\right) \cdot U_0^c \ .$$

Knowing B , the effective mass and inertia for the translational and rotational motion can be found in exactly the same manner as the earlier discussion by applying the appropriate boundary conditions. In particular it is to be noted that $G_i = 0 (i = 1 \text{ to } n)$ at 0_- and N_+ . A computer programme has also been written for the flexible cross member model. Numerical results will be given at the symposium.

Acknowledgments

The author wishes to thank his colleagues at the Royal Aircraft Establishment Messrs. F.C. Treble and N. Reed for their helpful discussions and to Mr. M.R. Gordon for the writing of the computer programmes.

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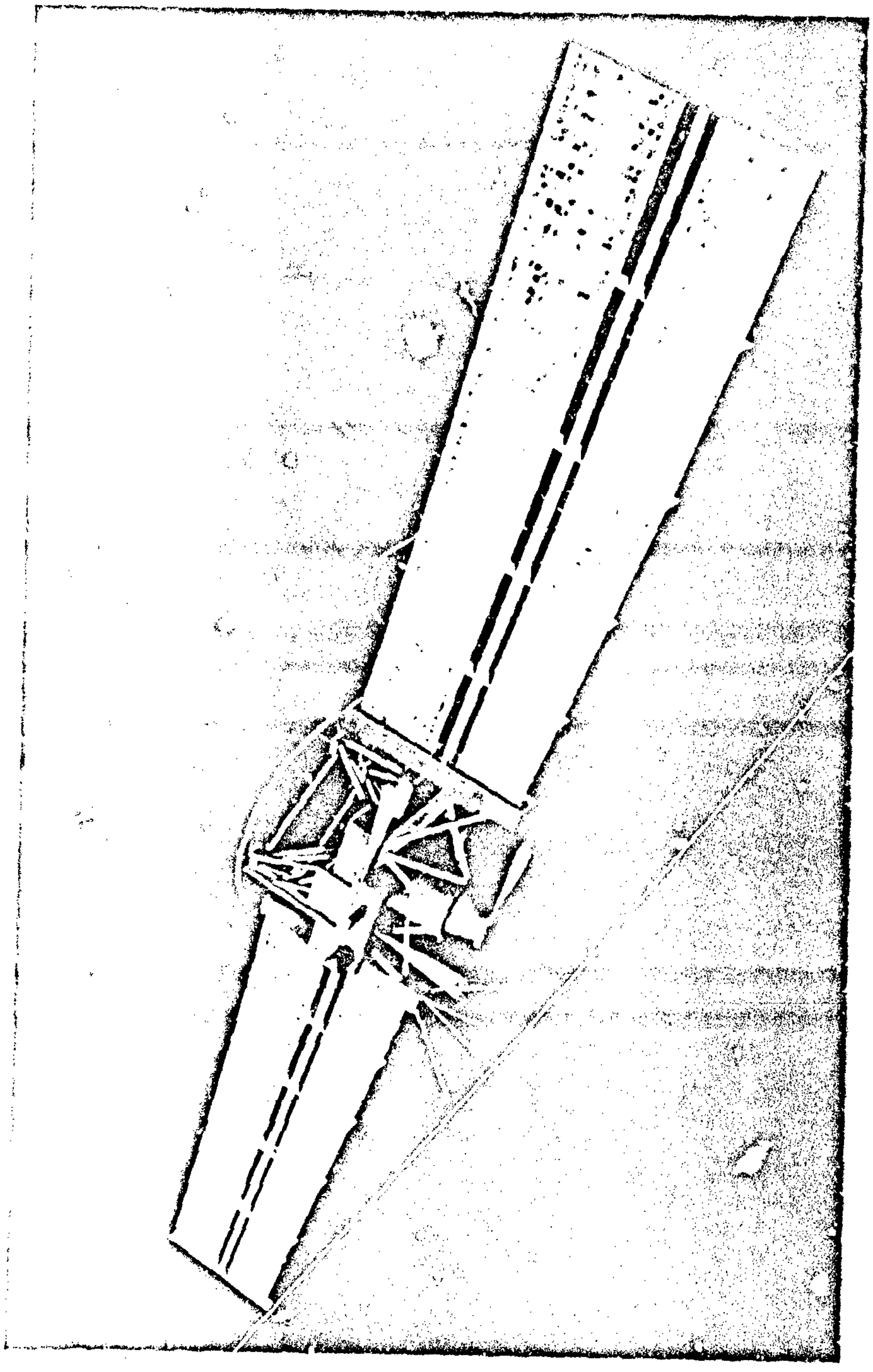

Fig.1 Spacecraft equipped with a two-paddle solar array of RAE design

Fig.2

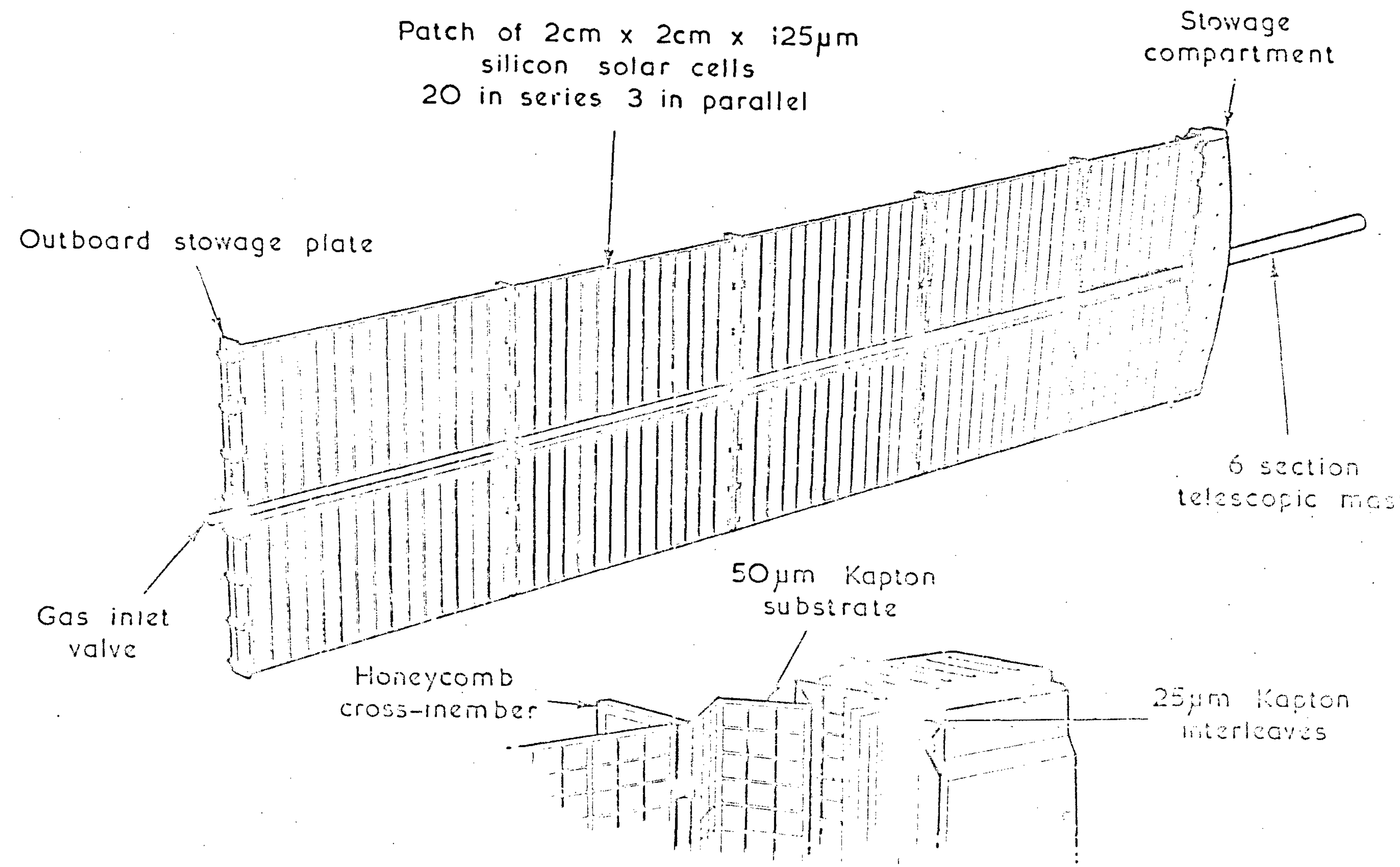

Fig.2 Experimental lightweight solar paddle

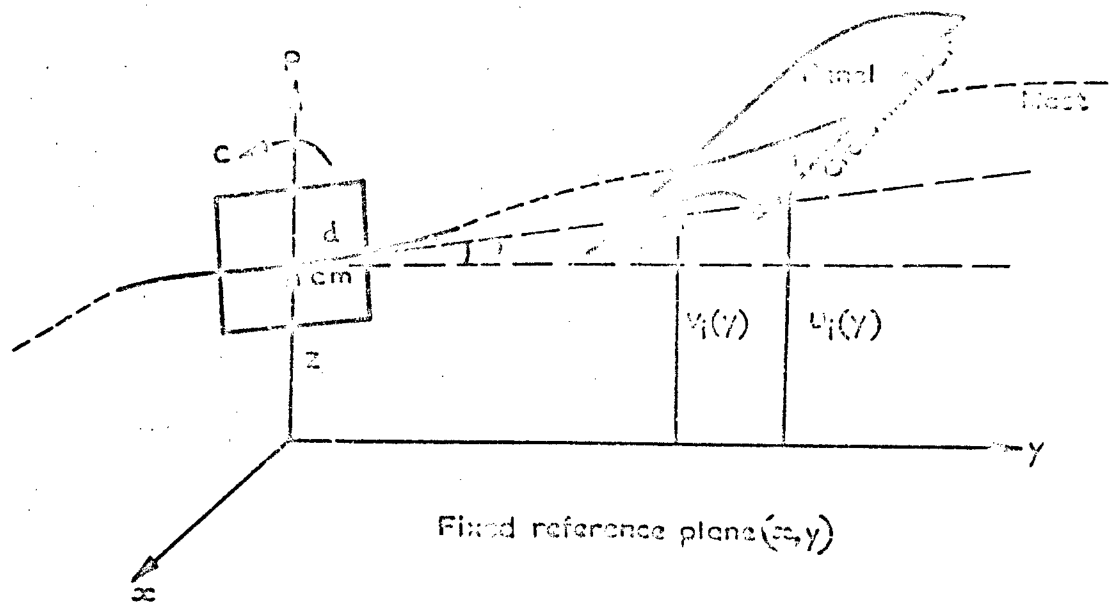

Fig.3 Coordinate system used for rigid cross member array

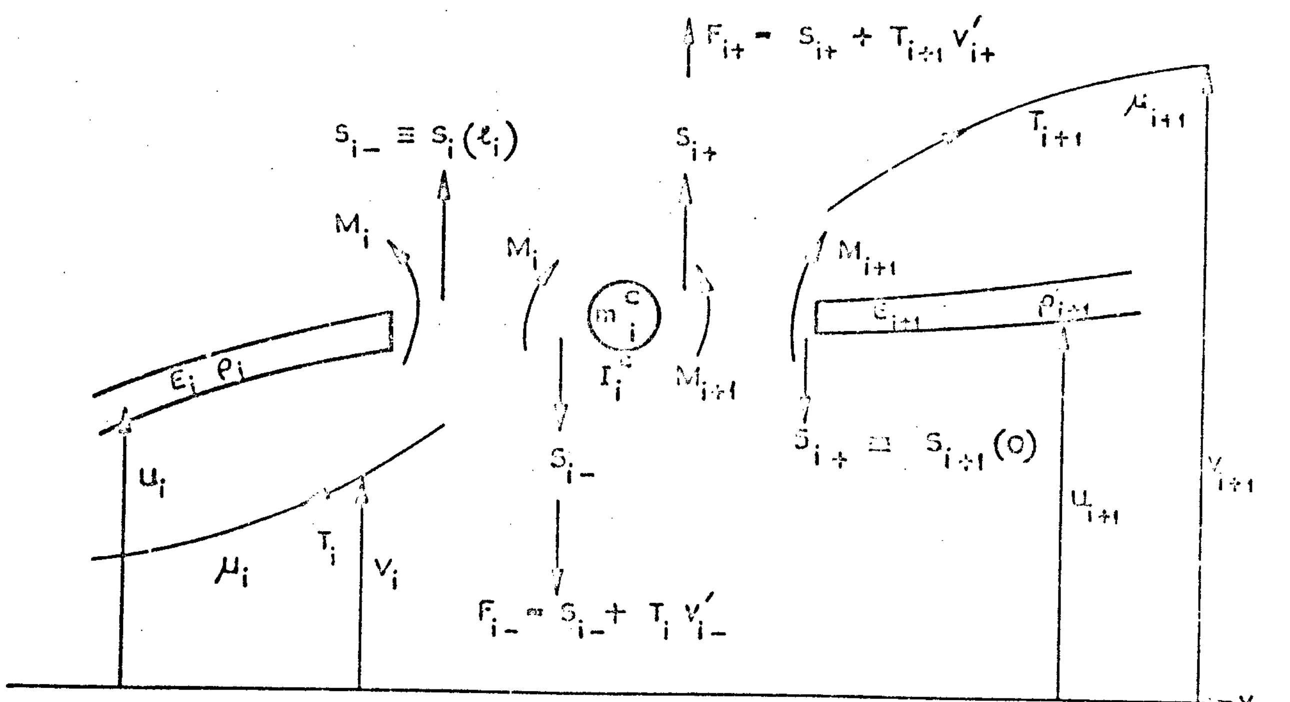

Fig.4 Forces and moments at i th cross member

Fig.5

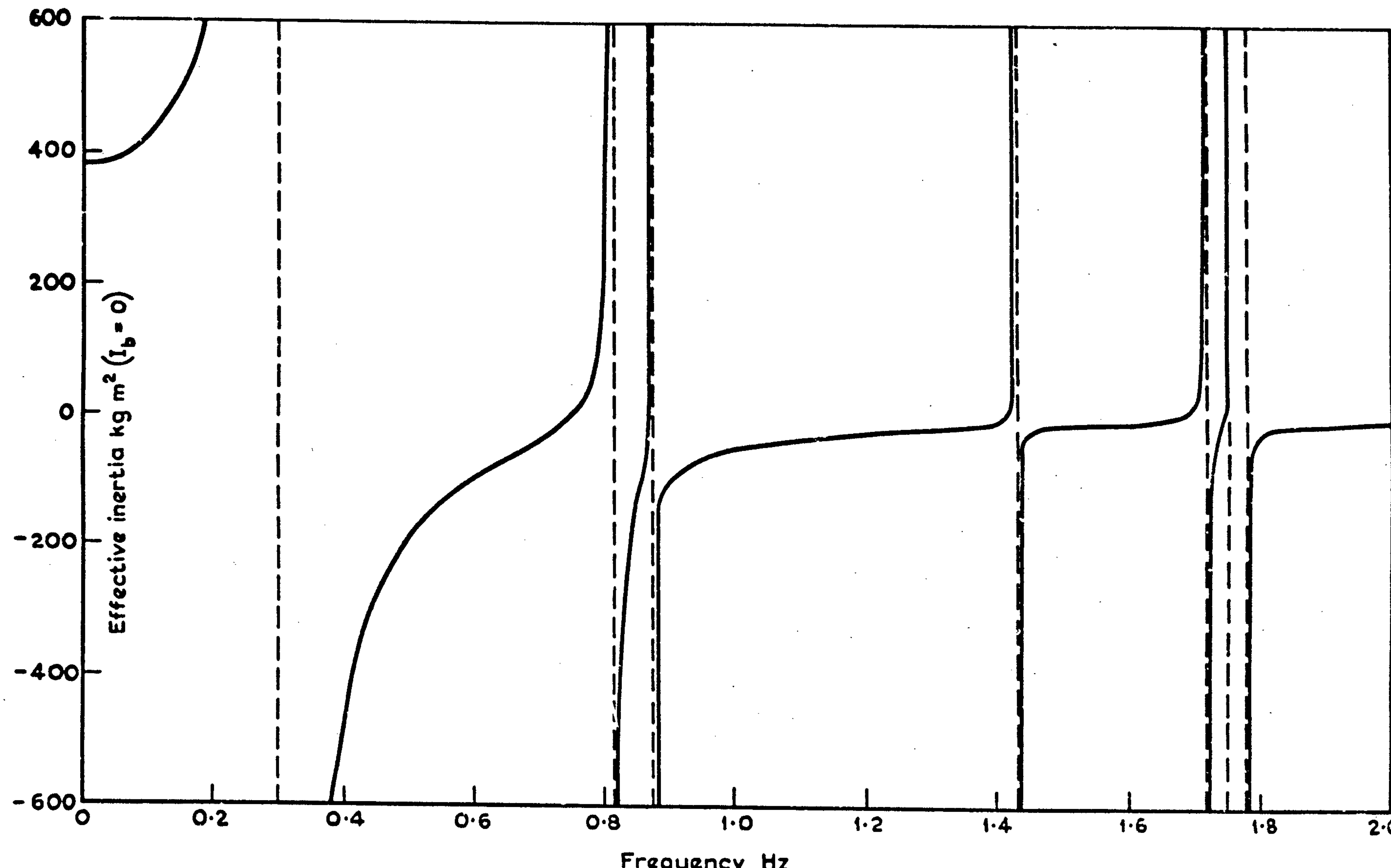

Fig.5 Dependence of effective inertia on frequency

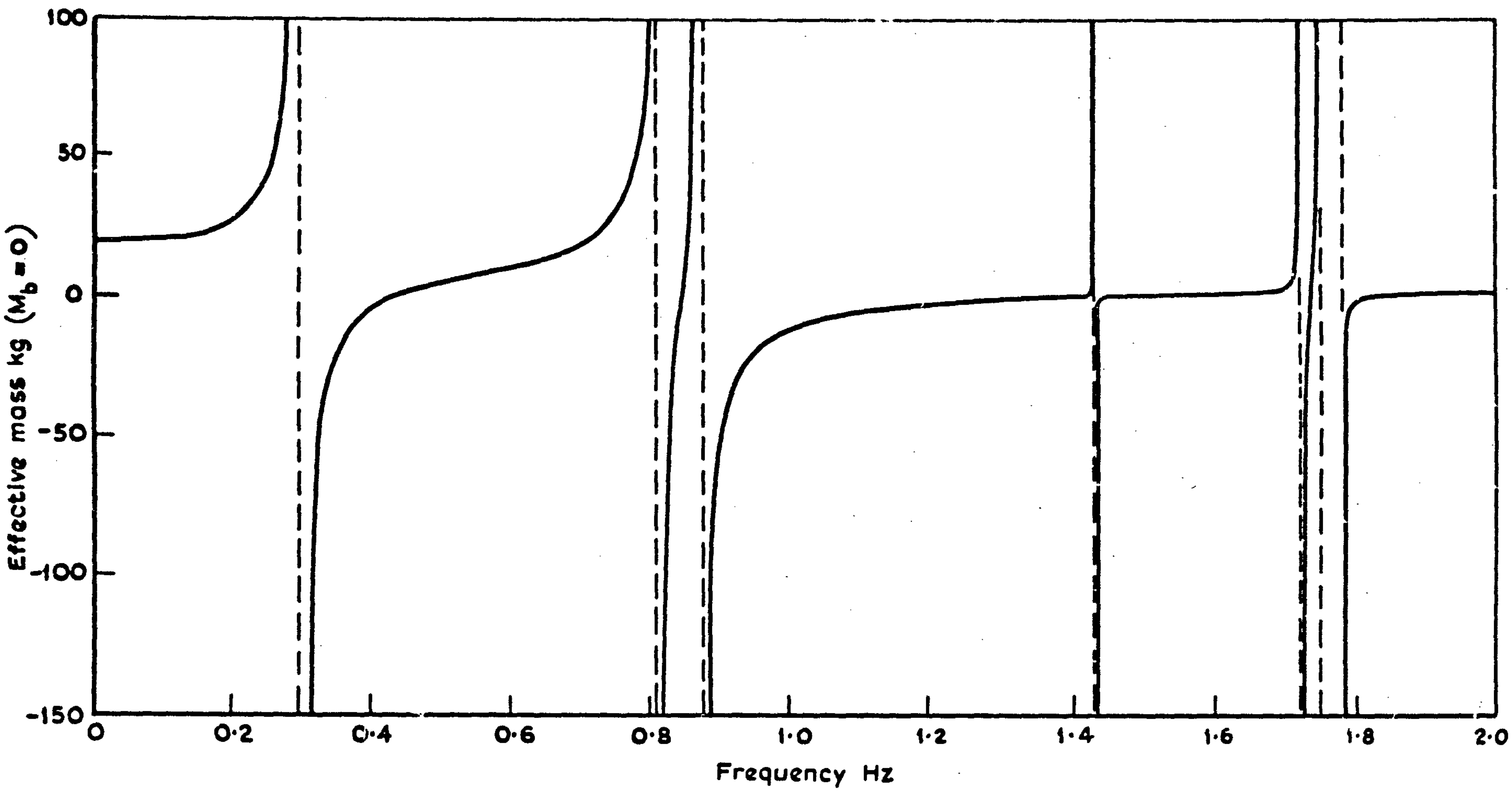

Fig.6 Dependence of effective mass on frequency

Fig.7

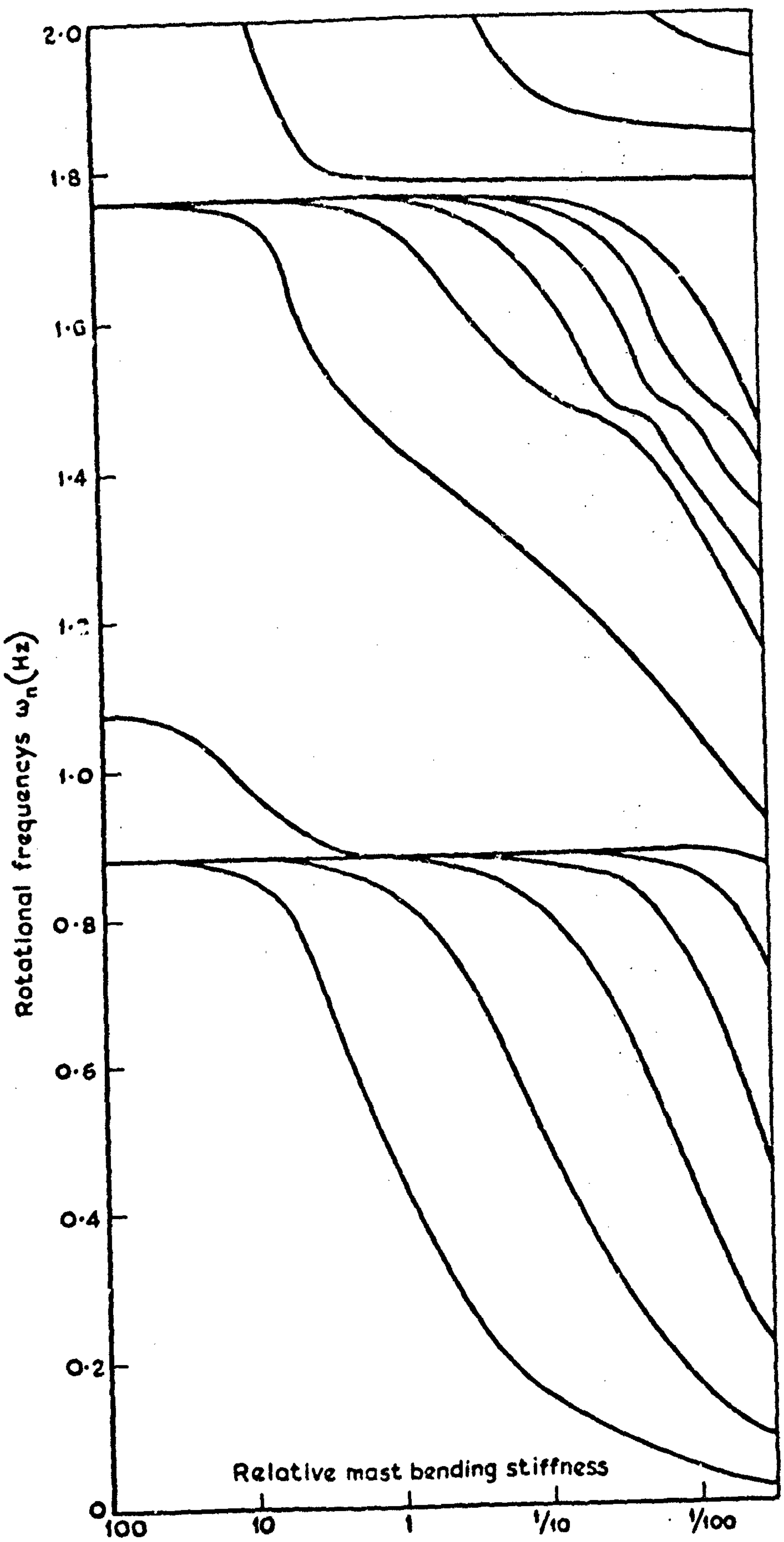

Fig.7 Variation of rotational frequencies with mast stiffness ($I_b = 300\ kg\,m^2$)

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